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Finite-size corrections to anomalous dimensions in $\mathcal{N} = 4$ SYM theory

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Abstract: The scaling dimensions of large operators in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory are dual to energies of semiclassical strings in $\mathrm{AdS}_5 \times S^5$. At one loop, the dimensions of large operators can be computed with the help of Bethe ansatz and can be directly compared to the string energies. We study finite-size corrections for Bethe states which should describe quantum corrections to energies of extended semiclassical strings.

Keywords: 1/N Expansion, AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

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1. Introduction

The semiclassical regime of the AdS/CFT correspondence [1, 2] relates classical string theory in $\mathrm{AdS}_5 \times S^5$ to perturbative $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory. This observation is quite remarkable and is at first glance counterintuitive, since the "Planck constant" of the AdS string theory $1/\sqrt{\lambda}$ is large at weak 't Hooft coupling $\lambda = g^2_{\mathrm{YM}} N$ in the SYM. The key point is that the true parameter of the semiclassical expansion for a large class of string states is not $1/\sqrt{\lambda}$ but $1/L$ [3], where L is a global charge of the string or some other large quantum number. At a given order in $1/L$, $1/\sqrt{\lambda}$ combines with L into the BMN coupling λ/L^2 [1] that can be large or small depending on L. It is customary now to identify the limit of small BMN coupling with the perturbative regime in the dual SYM theory.

The simplest quantities that can be directly compared to string theory are scaling dimensions of local operators. They are dual to energies of the string sates in $\mathrm{AdS}_5 \times S^5$. The loop corrections to scaling dimensions can be readily computed by standard diagrammatic techniques for simple operators, but to describe semiclassical string states we need operators with large quantum numbers. Straightforward diagrammatic calculations for such operators are hopeless because of the operator mixing which becomes more and more involved as the size of the operators grows. Fortunately, the problem simplifies considerably in the large-N limit when only planar diagrams contribute and can be reformulated in a useful way which makes it tractable. This reformulation also provides a nice physical interpretation of the operator mixing by identifying the mixing matrix with a hamiltonian of a certain one-dimensional spin system, which, quite remarkably, turns out to be integrable and exactly solvable [4]–[7]. The eigenvalues (anomalous dimensions) can then be calculated using the Bethe ansatz [8, 9] and compared to string energies. This has led to spectacular tests of the AdS/CFT correspondence for a variety of string solutions with large quantum numbers [10]–[27] and [3].

The relationship between classical strings and operators can be established at one and two loops quite generally, at the level of effective actions [19, 24, 26, 27] or by identifying integrable structures [23], which are present both in the spin chain [9] and on the string world-sheet [28]–[30]. The purpose of the present paper is to go beyond the leading order in $1/L$ in the SYM, which on the string side should correspond to taking into account quantum corrections. $1/L$ corrections to the point-like (BMN) string states [1] were studied in [31]–[33]. They can be compared to anomalous dimensions of near-BPS operators known exactly in the first few orders of perturbation theory [5]. The situation is more complicated for the extended string solutions. $1/L$ correction to the string energies are known for a particular class of solutions [12, 3], but the dual anomalous dimensions have been so far calculated only in the strict $L = \infty$ limit.

We shall consider the subsector of operators of bare dimension L made of two complex scalars: $Z = \Phi_1 + i\Phi_2$ and $W = \Phi_3 + i\Phi_4$:

$$\mathrm{tr}\, ZZZWWZWWZZZZ\ldots\,.$$

Linear combinations of such operators are in one-to-one correspondence with the states in the Hilbert space of a periodic spin-1/2 chain, where Z is identified with spin up and W with spin down on each site of the lattice:

$$|\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\downarrow\uparrow\uparrow\uparrow\uparrow\cdots\rangle\,.$$

The chain is periodic because of the cyclicity of the trace, which also imposes the condition of translation invariance (zero-momentum condition) on admissible states. The one-loop mixing matrix in this sector has a form of the Heisenberg hamiltonian [4]:

$$\Gamma = \frac{\lambda}{16\pi^2}\sum_{l=1}^{L}\left(1 - \sigma_l\cdot\sigma_{l+1}\right). \tag{1.1}$$

The higher-loop generalizations of this hamiltonian, potentially also integrable, have been discussed in the literature [5, 20] and [34]–[36]. The one-loop hamiltonian can be explicitly diagonalized by Bethe ansatz [8, 9]. The Bethe ansatz works also at two and three loops [20] and possibly extends to higher loop orders [36], but we will concentrate on the one-loop anomalous dimensions in this paper.[1]

The eigenstates of the Heisenberg hamiltonian with M down spins and with L and M large are dual to strings localized at the center of AdS and rotating on the five-sphere with two angular momenta $J_1 = L - M$ and $J_2 = M$. If the momenta satisfy

$$m_1 J_1 + m_2 J_2 = 0\,, \tag{1.2}$$

where m_1 and m_2 are integers, the classical solutions are very simple [37]. The angles on S^5 are then linear functions of the world-sheet coordinates. We shall concentrate on the gauge theory duals of these simplest uniform solutions.

[1]The Bethe ansatz of [20, 36] is only asymptotic and requires L to be large, but corrections to it seem to be exponentially small in the large L limit.

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2. Bethe ansatz

The vacuum of the hamiltonian (1.1) is an empty state with all spins up and corresponds to the chiral primary operator $\mathrm{tr}\, Z^L$. The operators of the form $\mathrm{tr}(Z^{L-M}W^M$+permutations) correspond to the states with M flipped spins and are characterizes by rapidities u_i of M magnons. The Bethe ansatz imposes a set of algebraic equations on the rapidities [8, 9]:

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k\neq j} \frac{u_j - u_k + i}{u_j - u_k - i}\,. \tag{2.1}$$

The cyclicity of the trace requires that the total momentum is zero or, in terms of rapidities, that

$$\prod_j \frac{u_j + i/2}{u_j - i/2} = 1\,. \tag{2.2}$$

The anomalous dimension is given by

$$\gamma = \frac{\lambda}{8\pi^2} \sum_i \frac{1}{u_i^2 + 1/4}\,. \tag{2.3}$$

These equations completely determine the spectrum of the hamiltonian (1.1).

We are interested in the limit when $L \to \infty$ and $M \to \infty$ and the filling fraction $\alpha = M/L$ is kept finite. Some particular solutions of Bethe equations in this scaling limit have been discussed in condensed matter literature [38, 39] and it is this limit that corresponds to the AdS duals of the semiclassical strings [13]. An inspection of the Bethe equations shows that Bethe roots scale with L as $u_i \sim L$. This allows us to simplify Bethe equations a bit. Writing them in the logarithmic form,

$$L \ln\left(\frac{u_j + i/2}{u_j - i/2}\right) = \sum_{k\neq j} \ln\left(\frac{u_j - u_k + i}{u_j - u_k - i}\right) - 2\pi n i\,, \tag{2.4}$$

and expanding in $1/u_i$, we get

$$\frac{L}{u_i} + 2\pi n = \sum_{j\neq i} \frac{2}{u_i - u_j}\,. \tag{2.5}$$

An arbitrary phase $2\pi n$ in (2.4) reflects the multivaluededness of the logarithm. In principle, different phases can be assigned to different Bethe roots, but a macroscopic number of roots should have the same phase to get a meaningful scaling limit at $L \to \infty$. The simplest case of equal phases corresponds to the SYM duals of the uniform string solutions [23]. We can indeed easily check that (1.2) is satisfied. Adding together the rescaled Bethe equations (2.5), we find the total momentum:

$$P = \sum_{i=1}^{M} \frac{1}{u_i} = -\frac{2\pi n M}{L}\,, \tag{2.6}$$

but (2.2) implies that the total momentum is an integer multiple of 2π, so

$$\alpha \equiv \frac{M}{L} = \frac{m}{n}\,, \tag{2.7}$$

which is equivalent to (1.2) if we put $m_1 = -m$, $m_2 = n - m$.

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Since Bethe equations are even in u_i, the corrections to (2.5) are of order $1/L^2$ and thus (2.5) is accurate up to and including $O(1/L)$. In the strict $L = \infty$ limit, (2.5) becomes a singular integral equation, which can be solved explicitly by the Riemann-Hilbert method [23] in the general case when the phases are different for different Bethe roots. If all phases are equal, the Riemann-Hilbert problem has an algebraic solution. As discussed in the appendix, the Riemann-Hilbert approach can then be generalized to take into account the leading $1/L$ correction. Here we use another method based on the observation of [40] (essentially present in the earlier work [41]–[44]) that the *exact* solution of (2.5) can be expressed in terms of the roots of associated Laguerre polynomials.

Let us briefly review the arguments of [40]. Consider the function defined as

$$Q(u) = \prod_{k=1}^{M} (u - u_k)\,, \tag{2.8}$$

where $\{u_i\}$ are the roots of (2.5). This function is known as the eigenvalue of the Baxter Q-operator [45]. The derivatives of Q evaluated at a root u_i, satisfy

$$u_i Q''(u_i) - (L + 2\pi n u_i) Q'(u_i) = 0$$

in virtue of (2.5), and thus the function $uQ''(u) - (L + 2\pi n u)Q'(u)$ is a polynomial of degree M which has the same roots $\{u_i\}$ as $Q(u)$. Hence, this function is just $Q(u)$, up to a coefficient. Comparing the u^M-terms one finds that

$$uQ''(u) - (L + 2\pi n u)Q'(u) + 2\pi n M Q(u) = 0\,. \tag{2.9}$$

The polynomial solution of this differential equation is the associated Laguerre polynomial: $Q(u) \propto L_M^{-(L+1)}(2\pi n u)$. So, the roots of the set of equations (2.5) are the roots of

$$L_M^{-(L+1)}(2\pi n u_i) = 0\,. \tag{2.10}$$

Some precautions have to be made here since the upper index of the associated Laguerre polynomial is negative. Using the Rodrigues representation and the fact that $L + 1 > M$ we find:

$$L_M^{-(L+1)}(x) = \frac{\mathrm{e}^{\,x} x^{L+1}}{M!} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^M e^{-x} x^{M-L-1} = \sum_{i=0}^{M} (-1)^M \binom{L-i}{M-i} \frac{x^i}{i!}\,. \tag{2.11}$$

We can now use this result to calculate the anomalous dimension in (2.3) using the relation

$$\sum_{i=1}^{M} \frac{1}{u_i^2} = \frac{1}{2\pi i} \oint_\Gamma \frac{\mathrm{d}\omega}{\omega^2} \frac{Q'(\omega)}{Q(\omega)}, \tag{2.12}$$

where Γ is a curve encircling all $\{u_i\}$ counterclockwise. We replaced $u_i^2 + 1/4$ by u_i^2, because $u_i = O(L)$ and we are interested only in the $1/L$ correction to the anomalous dimension. The only singularity outside of the contour of integration is a pole at $\omega = 0$. The contour can be deformed such that the integral picks up the residue:

$$\sum_i \frac{1}{u_i^2} = -\operatorname*{res}_{\omega=0} \left(\frac{Q'(\omega)}{\omega^2 Q(\omega)}\right) = -(2\pi n)^2 \left(\frac{L(0)L''(0) - (L'(0))^2}{(L(0))^2}\right). \tag{2.13}$$

Using the explicit form of the Laguerre polynomials (2.11), we find:

$$\frac{L(0)L''(0)-(L'(0))^2}{(L(0))^2}=\frac{M(M-1)}{L(L-1)}-\frac{M^2}{L^2}=\frac{\alpha(\alpha-1)}{L-1}\,, \tag{2.14}$$

where the definition of the filling fraction (2.7) was used. Finally, (2.3) yields:

$$\gamma=\frac{\lambda n^2\alpha(1-\alpha)}{2L}\left(1+\frac{1}{L}\right)+O\left(\frac{1}{L^3}\right)=\frac{\lambda m(n-m)}{2L}\left(1+\frac{1}{L}\right)+O\left(\frac{1}{L^3}\right). \tag{2.15}$$

This is our final result. The finite-size correction to the anomalous dimension turns out to be surprisingly simple and is just proportional to the leading order.

3. Discussion

We computed the finite-size correction to the energies of the simplest Bethe states that describe anomalous dimensions of large operators in $\mathcal{N}=4$ SYM. It would be interesting to calculate finite-size corrections to anomalous dimensions for a general Bethe state. Indeed, the scaling solution of the Bethe equations is known in complete generality and can be expressed in terms of abelian differentials on hyperelliptic Riemann surfaces [23]. The resolvent of Bethe roots plays the central role in this construction, and one may hope that calculations in the appendix can be reformulated in a similar geometric language and generalized to include arbitrary Bethe states. The calculations in the main text are based on another method, which reveals an interesting connection to the Baxter's Q-operator. The eigenvalues of the Q-operator are known to obey a finite difference equation equivalent to the set of Bethe equations [45]. On the other hand, the same eigenvalues satisfy a linear differential equation (2.9) in the scaling limit. We were not able to derive (2.9) directly from the Baxter equation, but it is possible that some relationship between them exists.

The string duals of Bethe states discussed in this paper are known at the classical level and are described by very simple string configurations in $\mathrm{AdS}_5\times S^5$. It would be very interesting to compute quantum corrections to their energies and to compare them to the $1/L$ corrections to anomalous dimensions calculated in this paper. However, the relevant string solutions are unstable [37], and it is not quite clear how to take into account quantum corrections for them.[2] This problem is rather puzzling, since the Bethe states at hand seem to be perfectly well-defined eigenstates of the Heisenberg hamiltonian and show no signs of instability. The resolution of this problem may provide further insights on the relationship between $\mathcal{N}=4$ SYM and strings.

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A. Finite-size corrections from loop equation

The scaling limit of the Bethe equations (2.5) has the same form as the saddle point equation for eigenvalues of a hermitean random matrix [46]. The analogy becomes complete if we rescale Bethe roots and define $x_i = u_i/L$, which is finite in the limit of $L \to \infty$. The rescaled variables satisfy

$$\frac{1}{x_i} + 2\pi n = \frac{2}{L}\sum_{j=1}^{M} \frac{1}{x_i - x_j} \tag{A.1}$$

and

$$P = \frac{1}{L}\sum_{j=1}^{M} \frac{1}{x_j} = -2\pi m \,. \tag{A.2}$$

An efficient way to solve matrix models, which turns out to be useful also in the present context, is to reformulate the problem in terms of the resolvent

$$G(x) = \frac{1}{L}\sum_{j=1}^{M} \frac{1}{x - x_j} \,. \tag{A.3}$$

The resolvent can be regarded as a generating function for eigenvalues of the commuting hamiltonians of the Heisenberg model [16, 17]:

$$P = -G(0)\,, \qquad \gamma = -\frac{\lambda}{8\pi^2 L}\, G'(0)\,, \qquad \ldots \tag{A.4}$$

In the matrix models the resolvent satisfies the loop equation [47, 48], an analog of which can be derived by multiplying both sides of (A.1) by $1/(x - x_i)$ and summing over i. Then

$$\begin{aligned}
\frac{2}{L^2}\sum_{i\neq j} \frac{1}{(x_i - x_j)(x - x_i)} &= \frac{1}{L^2}\sum_{i\neq j} \frac{1}{x_i - x_j}\left(\frac{1}{x - x_i} - \frac{1}{x - x_j}\right) \\
&= \frac{1}{L^2}\sum_{i\neq j} \frac{1}{(x - x_i)(x - x_j)} \\
&= \frac{1}{L^2}\sum_{ij} \frac{1}{(x - x_i)(x - x_j)} - \frac{1}{L^2}\sum_{i} \frac{1}{(x - x_i)^2} \\
&= G^2(x) + \frac{1}{L}G'(x).
\end{aligned}$$

Similar manipulations with the left-hand side give:

$$xG^2(x) + \frac{1}{L}xG'(x) = G(x) + 2\pi n x G(x) - 2\pi m\,, \tag{A.5}$$

where the momentum condition (A.2) in the form (A.4) was taken into account. We can now expand the resolvent in the powers of $1/L$:

$$G(x) = G_0(x) + \frac{1}{L}G_1(x) + O\left(\frac{1}{L^2}\right) \tag{A.6}$$

and plug this expansion in (A.5):

$$L^0 \ : 0 = xG_0^2 - (1+2\pi nx)G_0 + 2\pi m \tag{A.7}$$

$$L^{-1} \ : 0 = 2xG_0G_1 - (1+2\pi nx)G_1 + xG_0' \,. \tag{A.8}$$

The leading order is an algebraic equation whose solution is

$$G_0 = \frac{1}{2x}\left(1 + 2\pi nx - \sqrt{(1+2\pi nx)^2 - 8\pi n\alpha x}\right), \tag{A.9}$$

where (2.7) has been used to express m in terms of the filling fraction α. Alternatively, we could derive (2.7) from the loop equation by imposing the boundary condition $G(x) \to \alpha/x$ at infinity. The first correction can be found from equation (A.8):

$$\begin{aligned} G_1 &= \frac{xG_0'}{1+2\pi nx - 2xG_0} \\ &= \frac{1}{2x}\left[\frac{1+2\pi n(1-2\alpha)x}{(1+2\pi nx)^2 - 8\pi n\alpha x} - \frac{1}{\sqrt{(1+2\pi nx)^2 - 8\pi n\alpha x}}\right]. \end{aligned} \tag{A.10}$$

Thus, we have the resolvent up to order L^{-1}. Its Taylor expansion

$$G(x) = 2\pi n\alpha + (2\pi n)^2\alpha(\alpha - 1)\left(1 + \frac{1}{L}\right)x + \cdots \tag{A.11}$$

and (A.4) now yield

$$\gamma = \frac{\lambda n^2\alpha(1-\alpha)}{2L}\left(1+\frac{1}{L}\right) + \cdots,$$

in agreement with (2.15).

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